MV Calc, Ch 14A practice test  Good Luck to: __________________________  I ♥ MV Calc

Directions: You may not use a calculator. To receive any partial credit, show all pertinent work that supports your conclusions.

1.a) If \( \lim_{x \to x_0} f(x) = L \), where \( f(x) \) is a function of one variable, then the definition of \( \lim_{x \to a} f(x) = L \) is as follows: Definition 1: Given any number \( \epsilon > 0 \), we can find a \( \delta > 0 \) such that if \( |x - x_0| < \delta \), then \( |f(x) - L| < \epsilon \). Use the definition above to prove that \( \lim_{x \to 2} \frac{2}{3} x + 1 = \frac{7}{3} \).

b) Recall the definition of a limit on a point of a surface. Let \( f(x, y) \) be a function of two variables, and assume that \( f \) is defined at all points of some open disk centered at \( (x_0, y_0) \), except possibly at \( (x_0, y_0) \).

We define \( \lim_{(x,y) \to (x_0,y_0)} f(x,y) = L \) if given any number \( \epsilon > 0 \), we can find a number \( \delta > 0 \) such that \( f(x,y) \) satisfies \( |f(x,y) - L| < \epsilon \) whenever the distance between \( (x, y) \) and \( (x_0, y_0) \) satisfies \( 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \). Use the diagram shown above to explain how the diagram conveys the above definition 2.

2. a) Show that the value of \( \frac{xyz}{x^2 + y^2 + z^2} \) approaches 0 as \( (x, y, z) \to (0,0,0) \) along any line \( x = at \), \( y = bt \), \( z = ct \).

b) Show that the limit \( \lim_{(x,y,z) \to (0,0,0)} \frac{xyz}{x^2 + y^4 + z^4} \) does not exist by letting \( (x, y, z) \to (0,0,0) \) along the curve \( x = t^2 \), \( y = t \), \( z = t \).

<table>
<thead>
<tr>
<th>Conc c (in ppm)</th>
<th>Time t in months</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>14</td>
</tr>
<tr>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
</tr>
<tr>
<td>6</td>
<td>96</td>
</tr>
<tr>
<td>15</td>
<td>96</td>
</tr>
</tbody>
</table>

3. An experiment to measure the toxicity of formaldehyde yielded the data in the table shown above. The values show the percent, \( P = f(t, c) \), of rats surviving an exposure to formaldehyde at a concentration of \( c \) (in parts per million, ppm) after \( t \) months. Estimate \( f_t(18, 6) \) and \( f_c(18, 6) \). Interpret your answers in terms of formaldehyde toxicity.

4. In each case, give a possible contour diagram for the function \( f(x, y) \) if
   - a) \( f_x > 0 \) and \( f_y > 0 \)
   - b) \( f_x > 0 \) and \( f_y < 0 \)
   - c) \( f_x < 0 \) and \( f_y > 0 \)
   - d) \( f_x < 0 \) and \( f_y < 0 \)

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5. A contour diagram for \( z = f(x, y) \) is shown above.

a) Is \( \frac{\partial z}{\partial x} \) positive or negative? Explain your response.

b) Is \( \frac{\partial z}{\partial y} \) positive or negative? Explain your response.

c) Estimate \( f(2,1), \frac{\partial z}{\partial x}(2,1) \), and \( \frac{\partial z}{\partial y}(2,1) \).

6. The Laplace Equation is known as \( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \). Verify that \( z = e^x \sin y + e^y \cos x \) satisfies Laplace’s Equation.
Directions: You may not use a calculator. To receive any partial credit, show all pertinent work that supports your conclusions.

1.a) If \( \lim_{x \to x_0} f(x) = L \), where \( f(x) \) is a function of one variable, then the definition of \( \lim_{x \to x_0} f(x) = L \) is as follows: Definition 1: Given any number \( \epsilon > 0 \), we can find a \( \delta > 0 \) such that if \( |x - x_0| < \delta \), then \( |f(x) - L| < \epsilon \). Use the definition above to prove that \( \lim_{x \to 2} \frac{2}{3}x + 1 = \frac{7}{3} \).

Solution: Let \( \epsilon > 0 \). Choose \( \delta = \frac{3}{2} \epsilon \). Suppose that \( |x - 2| < \delta \Rightarrow \frac{2}{3}x + 1 < \frac{7}{3} \epsilon \).

\[ |f(x) - L| < \epsilon. \]

b) Recall the definition of a limit on a point of a surface. Let \( f(x, y) \) be a function of two variables, and assume that \( f \) is defined at all points of some open disk centered at \((x_0, y_0)\), except possibly at \((x_0, y_0)\).

We define \( \lim_{(x, y) \to (x_0, y_0)} f(x, y) = L \) if given any number \( \epsilon > 0 \), we can find a number \( \delta > 0 \) such that \( f(x, y) \) satisfies \( |f(x, y) - L| < \epsilon \) whenever the distance between \((x, y)\) and \((x_0, y_0)\) satisfies

\[ 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta. \]

Use the diagram shown above to explain how the diagram conveys the above definition 2.

Solution: From your text see page 939.

2. a) Show that the value of \( \frac{xyz}{x^2 + y^2 + z^2} \) approaches 0 as \((x, y, z) \to (0, 0, 0)\) along any line \( x = at \), \( y = bt \), \( z = ct \).

\[ \lim_{t \to 0} \frac{atbtct}{a^2t^2 + b^4t^4 + c^4t^4} = \lim_{t \to 0} \frac{abct^3}{a^2t^3 + b^4t^4 + c^4t^4} = \lim_{t \to 0} \frac{abct}{a^2 + b^4t^2 + c^4t^2} = 0 \]

Solution:

b) Show that the limit \( \lim_{(x, y, z) \to (0, 0, 0)} \frac{xyz}{x^2 + y^2 + z^2} \) does not exist by letting \((x, y, z) \to (0, 0, 0)\) along the curve \( x = t^2, \ y = t, \ z = t \).

\[ \lim_{t \to 0} \frac{t^2 \cdot t \cdot t}{t^4 + t^4 + t^4} = \lim_{t \to 0} \frac{t^2 \cdot t \cdot t}{t^4 + t^4 + t^4} = \frac{1}{3}. \]

From parts a and b we can see that along two different parametric paths two different limits were obtained. Therefore, \( \lim_{(x, y, z) \to (0, 0, 0)} \frac{xyz}{x^2 + y^2 + z^2} \) does not exist.
3. An experiment to measure the toxicity of formaldehyde yielded the data in the table shown above. The values show the percent, \( P = f(t, c) \), of rats surviving an exposure to formaldehyde at a concentration of \( c \) (in parts per million, ppm) after \( t \) months. Estimate \( f(t, 6) \) and \( f_c(t, 6) \). Interpret your answers in terms of formaldehyde toxicity.

Solution: We have

\[
f_t(18, 6) = \frac{\Delta P}{\Delta t} = \frac{90 - 93}{20 - 18} = -1.5 \text{ percent per month.}
\]

Eighteen months after rats are exposed to a formaldehyde concentration of 6 ppm, the percent of rats surviving is decreasing at a rate of about 1.5 percent per month. In other words, during the eighteenth month, an additional 1.5% of the rats die (how morbid).

We have

\[
f_c(18, 6) = \frac{\Delta P}{\Delta C} = \frac{82 - 93}{15 - 6} = -1.22 \text{ percent /ppm.}
\]

If the original concentration increases by 1 ppm, the percent surviving after 18 months decreases by about 1.22.

4. In each case, give a possible contour diagram for the function \( f(x, y) \) if

a) \( f_x > 0 \) and \( f_y > 0 \)  
b) \( f_x > 0 \) and \( f_y < 0 \)  
c) \( f_x < 0 \) and \( f_y > 0 \)  
d) \( f_x < 0 \) and \( f_y < 0 \)

Solution:

a) Since \( f_x > 0 \), the values on the contours increase as you move to the right. Since \( f_y > 0 \), the values on the contours increase as you move upward. See figure below.

![Figure 14.1](image1.png)

Figure 14.1: \( f_x > 0 \) and \( f_y > 0 \)

b) Since \( f_x > 0 \), the values on the contours increase as you move to the right. Since \( f_y < 0 \), the values on the contours decrease as you move upward. See figure below.

![Figure 14.2](image2.png)

Figure 14.2: \( f_x > 0 \) and \( f_y < 0 \)
c) Since \( f_x < 0 \), the values on the contours decrease as you move to the right. Since \( f_y > 0 \), the values on the contours increase as you move upward. See figure below.

\[ f_x < 0 \text{ and } f_y > 0 \]

\[ f_x < 0 \text{ and } f_y < 0 \]

5. A contour diagram for \( z = f(x, y) \) is shown above.

a) Is \( \frac{\partial z}{\partial x} \) positive or negative? Explain your response.

Solution: \( \frac{\partial z}{\partial x} > 0 \) because as \( x \) increases the level curves are increasing.

b) Is \( \frac{\partial z}{\partial y} \) positive or negative? Explain your response.

Solution: \( \frac{\partial z}{\partial y} < 0 \) because as \( y \) increases the level curves are decreasing.

c) Estimate \( f(2,1) \), \( \left. \frac{\partial z}{\partial x} \right|_{(2,1)} \), and \( \left. \frac{\partial z}{\partial y} \right|_{(2,1)} \).

\[ f(2,1) \approx 10, \quad \left. \frac{\partial z}{\partial x} \right|_{(2,1)} \approx \frac{4}{2} = 2, \quad \left. \frac{\partial z}{\partial y} \right|_{(2,1)} \approx -4 \]
6. The Laplace Equation is known as \( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \). Verify that \( z = e^x \sin y + e^y \cos x \) satisfies Laplace's Equation.

Solution: \( \frac{\partial z}{\partial x} = e^x \sin y - e^y \sin x \), \( \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left( e^x \sin y - e^y \sin x \right) = e^x \sin y - e^y \cos x \)

\( \frac{\partial z}{\partial y} = e^x \cos y + e^y \cos x \), \( \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} \left( e^x \cos y + e^y \cos x \right) = -e^x \sin y + e^y \cos x \)

\( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^x \sin y - e^y \cos x - e^x \sin y + e^y \cos x = 0 \)